Journal of Statistical Physics, Vol. 105, Nos. 5/6, December 2001 (© 2001)

Hausdorff Dimension of Non-Hyperbolic Repellers. I: Maps with Holes

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Received March 15, 2001

This is the first paper in a two-part series devoted to studying the Hausdorff dimension of invariant sets of non-uniformly hyperbolic, non-conformal maps. Here we consider a general abstract model, that we call piecewise smooth maps with holes. We show that the Hausdorff dimension of the repeller is strictly less than the dimension of the ambient manifold. Our approach also provides information on escape rates and dynamical dimension of the repeller.

KEY WORDS: Hausdorff dimension; non-uniform hyperbolicity; repeller; dynamical dimension.

1. INTRODUCTION

This work was originally motivated by the following problem. Suppose g is a globally hyperbolic (Anosov) diffeomorphism, in dimension 3 or higher, and p is some fixed point of g with two expanding eigenvalues. Let g go through a Hopf bifurcation, so that the saddle point p becomes an attractor. See Fig. 1. The complement Λ of the basin of attraction $W^s(p)$ is a repeller for the new diffeomorphism. *Does Λ have zero Lebesgue measure (volume)?* Even more, *is the Hausdorff dimension of the repeller strictly less than the dimension of the ambient manifold?*

Fractals invariants such as the Hausdorff dimension play an important role in various areas of Dynamical Systems, and have attracted a great deal of attention. We refer the reader to refs. 5, 7, and 9 for an updated panorama of the theory. Computing these fractal invariants is usually difficult, because they depend on the microscopic structure of the set. Not

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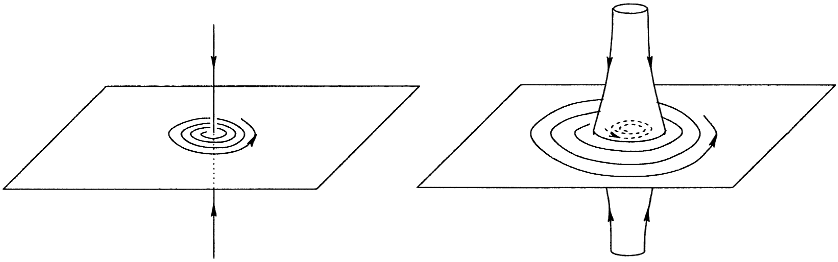

Fig. 1. A Hopf bifurcation.

surprisingly, most methods require the set to be self-similar, meaning that small pieces of it look very much like the whole. And self-similarity often arises from the dynamical system being uniformly hyperbolic (contracting and/or expanding) and conformal, possibly, after some dimension reduction.

As it turns it out, neither of these properties holds in the setting that we mentioned before. On the one hand, the repeller contains an invariant circle that is produced by the Hopf bifurcation, and so it can never be hyperbolic. On the other hand, conformality being a non-generic property, in most cases these diffeomorphisms are not conformal, nor can they be reduced to conformal maps. Nevertheless, we are able to give a positive answer to the questions raised above: *the Hausdorff dimension of the repeller is strictly less than the dimension of the ambient manifold;* in particular, Λ has volume zero. This is proved in ref. 6, as an application of the results we obtain in the present paper.

Here we deal with a general abstract setting that, in particular, models the behaviour of those diffeomorphisms along the central (non-hyperbolic) direction. This abstract model is described by finite-to-one piecewise smooth maps f on a manifold of any dimension $d \geqslant 1$, sending a domain V onto a larger one $W \supset V$. The repeller is the set of points in V whose forward orbits never fall into the "hole" $H = W \setminus V$. See Fig. 2. We prove that *if the map is non-uniformly expanding,* in a sense that will be made precise later (implying positive Lyapunov exponents Lebesgue almost everywhere), *then the Hausdorff dimension of the repeller is strictly less than d*. The precise statement will appear in Theorem 1 later.

Ideas involved in the proof are quite general, and we expect them to be useful in other situations. To handle the fact that our maps are not uniformly expanding, we construct a new dynamical system, induced from the original one, which has properties of uniform expansion and bounded distortion. This induced map is defined on a large subset: the complement has small Hausdorff dimension. To go around non-conformality, we prove that

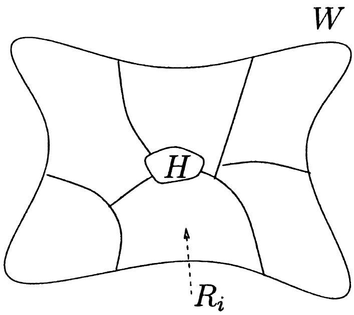

Fig. 2. A piecewise smooth map with holes.

volume estimates obtained from the bounded distortion property can be turned into diameter estimates. This allows us to get the results we stated, as well as further geometric information about the repeller.

1.1. Non-Conformal Maps with Holes

Here we describe our abstract model, and state the main result of this paper. Let $f: M \to M$ be a map on a d-dimensional Riemannian manifold, $d \geqslant 1$ such that

(A_1) There exist domains $R_1, \ldots, R_m$ in M, whose interiors are two-by-two disjoint, such that the restriction of f to each R_i is a $C^{1+\varepsilon}$ diffeomorphism onto some domain W_i that contains $R_1 \cup \cdots \cup R_m$. The difference $H_i = W_i \setminus (R_1 \cup \cdots \cup R_m)$ has non-empty interior, and the boundaries $\partial R_1, \ldots, \partial R_m$ have limit capacity less than d.

Figure 2 describes an example where $H_i = H$ and $W_i = W$ are the same for all i. By *domain* we mean a compact path-connected subset. The smoothness requirement above means that $f \mid R_i$ is a C^1 diffeomorphism, in the sense of Whitney, with ε-Hölder continuous Jacobian det Df. The *limit capacity,* or *box dimension,* of a metric space X is defined by

$$c(X) = \limsup_{\varepsilon \to 0} \frac{\log n(X, \varepsilon)}{|\log \varepsilon|},$$

where $n(X, \varepsilon)$ is the smallest number of ε-balls needed to cover X.

Let f be as in (A_1). The *repeller* of f in $R_1 \cup \cdots \cup R_m$ is the set of points Λ whose forward orbits never fall into the H_i, that is,

$$\Lambda = \{x: f^n(x) \in R_1 \cup \cdots \cup R_m \text{ for every } n \geqslant 0\}.$$

See Fig. 3.

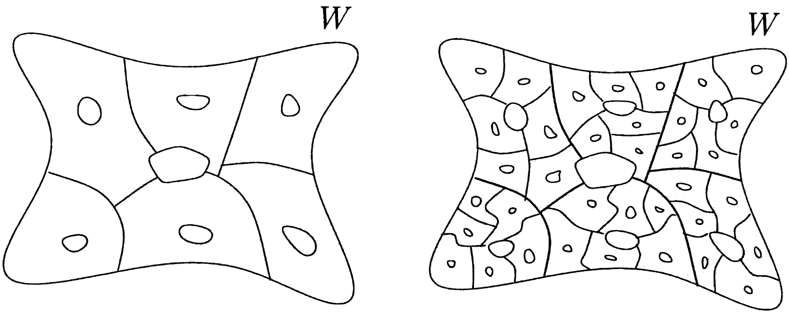

Fig. 3. The repeller of a piecewise smooth map with holes.

Given $n \geqslant 1$, we call *n-cylinder* any set of the form

$$C(\alpha_1,\ldots, \alpha_n) = R_{\alpha_1} \cap f^{-1}(R_{\alpha_2}) \cap \cdots \cap f^{-n+1}(R_{\alpha_n})$$

with $\alpha_1, \alpha_2,\ldots, \alpha_n$ in $\{1,\ldots, m\}$. That is, an n-cylinder consists of all the points remaining in $R_1 \cup \cdots \cup R_m$, and sharing a given itinerary with respect to the family $\{R_1,\ldots, R_m\}$, up to time n. Clearly, n-cylinders form a covering of the repeller Λ, for each $n \geqslant 1$.

For each $n \geqslant 1$ and $\alpha_1, \alpha_2,\ldots, \alpha_n$ in $\{1,\ldots, m\}$, we consider the *average least expansion*

$$\phi_n(\alpha_1, \alpha_2,\ldots, \alpha_n) = \frac{1}{n} \sum_{j=1}^{n} \inf_{x \in C_j} \log \|Df^{-1}(f^j(x))\|^{-1},$$

where the infimum is taken over all x in $C_j = C(\alpha_1,\ldots, \alpha_j)$. Throughout, $Df^{-i}(f^j(y))$ is to be understood as the inverse of the derivative $Df^i(f^{j-i}(y))$, for any y and $j \geqslant i$. Note that $\phi_n(\alpha_1, \alpha_2,\ldots, \alpha_n) > c > 0$ implies that the derivative Df^n expands every vector:

$$\|Df^{-n}(f^n(x))\| \leqslant \prod_{j=1}^{n} \|Df^{-1}(f^j(x))\| \leqslant e^{-cn} \quad \text{for all } x \text{ in } C(\alpha_1,\ldots, \alpha_n).$$

We also assume that

(A_2) There exist $c > 0$ and $c_1 > 0$ such that, for every large n, we have $\phi_n(\alpha_1, \alpha_2,\ldots, \alpha_n) > c$ except on a subset $\mathcal{Q}_n$ of n-cylinders whose total volume decreases exponentially fast with time: $\sum_{C \in \mathcal{Q}_n} \mathrm{Leb}(C) \leqslant e^{-c_1 n}$.

For $\alpha \geqslant 0$, the Hausdorff α-measure of a metric space X is defined by

$$m_\alpha(X) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_{U \in \mathscr{U}} (\operatorname{diam} U)^\alpha : \mathscr{U} \text{ is an open covering of } X \text{ with } \operatorname{diam} U \leqslant \varepsilon \text{ for all } U \in \mathscr{U} \right\}.$$

It is easy to show that there exists a unique real number $\mathrm{HD}(X)$, called *Hausdorff dimension of* X, such that $m_\alpha(X) = \infty$ for any $\alpha < \mathrm{HD}(X)$ and $m_\alpha(X) = 0$ for any $\alpha > \mathrm{HD}(X)$.

Theorem 1. Let $f: M \to M$ and Λ be as above, satisfying (A_1) and (A_2). Then $\mathrm{HD}(\Lambda) < d$.

A few comments are in order, concerning our hypotheses. On the one hand, as we shall see in Section 2, the condition about the boundaries in (A_1) can be relaxed: it suffices to assume that each restriction $f \mid R_i$ can be extended to some larger domain R_i' whose boundary has limit capacity less than d. The interiors of these larger domains need not be two-by-two disjoint.

On the other hand, some control of the rate of decay as we assumed in (A_2) is indeed necessary for the conclusion, as we explain at the end of Section 1.2. The example that we give there suggests that a summability condition might be enough, and this is so in some special situations; see Remark 2. However, for the general case in dimension $d > 1$, our arguments in Section 4 currently require exponential decay.

Remark 1. The conditions that W_i contain $R_1 \cup \cdots \cup R_m$ and H_i have non-empty interior, for every $1 \leqslant i \leqslant m$, are too strong. More generally, we may suppose that the intersection of W_i with $R_1 \cup \cdots \cup R_m$ coincides with $\bigcup_{j \in J(i)} R_j$ for some subset $J(i)$ of $\{1,..., m\}$. Moreover, for every i there exists $\ell \geqslant 0$ such that $f^\ell(R_i)$ contains some R_k whose $H_k = W_k \setminus \bigcup_{j \in J(k)} R_j$ has non-empty interior. The repeller is the set of points that do not fall into the H_i at any iterate: for any $n \geqslant 0$ there is $i \in \{1,..., m\}$ such that $f^n(x) \in R_i$ and $f^{n+1}(i) \in \bigcup_{j \in J(i)} R_j$. Our arguments extend to this situation, in a straightforward way, to prove that the Hausdorff dimension of the repeller is less than d.

The proof of Theorem 1 occupies Sections 2, 3, and 4. In passing, we obtain other results about rates of escape and dynamical dimension of the repeller. See Section 3 for definitions and statements. Right now, let us close this Introduction with an outline of the proof.

1.2. Motivations and Ideas of the Proof

In order to establish an upper estimate α for the Hausdorff dimension of a subset of a metric space it is enough to exhibit a sequence $\mathcal{U}_n$ of coverings with diameter going to zero, and whose Hausdorff α-measures are uniformly bounded: there exists $M > 0$ so that

$$H_\alpha(\mathcal{U}_n) = \sum_{U \in \mathcal{U}_n} \operatorname{diam}(U)^\alpha \leqslant M \qquad \text{for every } n. \tag{1}$$

For instance, in the case of the mid-third Cantor set such a sequence can be constructed along the following lines: at each step some interval U in $\mathcal{U}_n$ is replaced by two subintervals, so that the α-measure of the new covering $\mathcal{U}_{n+1}$ is smaller than that of $\mathcal{U}_n$, for fixed $\alpha < 1$ close enough to 1. This is made possible by the key fact that a sizable portion of U corresponds to some gap of the Cantor set. More generally, a similar argument proves that any dynamically defined Cantor set in the real line, in the sense of ref. 7, has Hausdorff dimension bounded by some $\alpha < 1$: a bounded distortion property ensures the key fact mentioned before, for intervals in all size scales.

For proving Theorem 1 we try to use a similar strategy. One important point is that we must start by basing our construction on the notion of volume rather than diameter. Of course, for intervals the two notions coincide, which is one of the things that makes the one-dimensional situation discussed above much easier. More precisely, we aim at constructing a sequence $\mathcal{U}_n$ of coverings such that $\operatorname{diam}(\mathcal{U}_n) \to 0$ and

$$\sum_{U \in \mathcal{U}_n} \operatorname{Leb}(U)^\beta \leqslant K, \qquad \text{for every } n, \tag{2}$$

where $K > 0$ and $\beta < 1$ are constants independent on n. As a matter of fact, the $\mathcal{U}_n$ may fail to cover some small subset of the repeller, negligible for our purposes, as we shall explain in a little while.

The construction of these coverings is by successive refinement: we obtain $\mathcal{U}_{n+1}$ by replacing each element U of $\mathcal{U}_n$ by sub-domains U_j, such that

$$\sum_j \operatorname{Leb}(U_j)^\beta \leqslant \operatorname{Leb}(U)^\beta. \tag{3}$$

That this is possible, comes from the fact that U contains some hole of the repeller, i.e., a pre-image of H_i by an iterate of f, that contains a sizable part of U (but see also the remarks at the end of this section). Now, to have this last property at every stage of the construction, we need a property of bounded distortion for the Jacobian, and that is one of the main difficulties of our problem: due to the lack of hyperbolicity we can not expect our maps to have such a property.

To solve this difficulty, in Section 2 we construct a new map, obtained from the original f by *inducing*: roughly speaking, we iterate f a convenient number of times (varying with the point), so as to make the derivative expand uniformly. This new map F is piecewise smooth and expanding on each smoothness domain. Most important for our purposes, the Jacobian does satisfy a bounded distortion condition. The key for constructing F is the property of non-uniform expansion (A_2). One of the first steps is to show that the exceptional set where the condition in (A_2) fails has Hausdorff dimension strictly less than d, and so may be neglected for all our purposes.

We use this induced map F to define our coverings $\mathscr{U}_n$: in brief terms, each U in $\mathscr{U}_n$ is an n-cylinder for F. The fact that $\operatorname{diam}(\mathscr{U}_n) \to 0$ is a consequence of the expansiveness of F, whereas (2) follows from bounded distortion. At this point we are already able to prove that the repeller has *non-zero rate of escape* for F, that is, the volume of the set of points remaining within any small distance from Λ decreases exponentially fast with time. This is done in Section 3.

However, (2) is still insufficient for estimating the Hausdorff dimension of the repeller. The reason is that the latter notion is defined in terms of diameters of covering sets, rather than volumes. As we explained before, this difficulty is typical of higher dimensional (non-conformal) situations. To solve it, we bring in another main idea: we prove in Section 4 that the elements of the $\mathscr{U}_n$ may be covered by *balls* in such a way that the new covering $\mathscr{B}_n$ thus obtained satisfies

$$\sum_{B \in \mathscr{B}_n} \operatorname{Leb}(B)^\gamma \leqslant C \sum_{U \in \mathscr{U}_n} \operatorname{Leb}(U)^\beta \leqslant CK \tag{4}$$

for some $C > 0$ and $\gamma \in (\beta, 1)$ independent of n. This is done in Proposition 4.1, which is really rather general. It is at this point that the condition on the limit capacity of the boundary is used.

Since the volume of a ball is closely related to its diameter, the last inequality (4) immediately yields a bound like (1) for the coverings $\mathscr{B}_n$, with exponent $\gamma d < d$. In this way we prove that the subset of the repeller contained in the domain of F has Hausdorff dimension strictly less than d. We already mentioned that the Hausdorff dimension of the complement is also less than d, so Theorem 1 follows.

Control of the rate of decay as assumed in (A_2) is used at a few steps, e.g., for inequality (2). Let us point out that some such control is necessary for the result itself: the following example shows that if one drops the assumption that $\mathrm{Leb}(U_j)$ decays relatively fast, then the Hausdorff dimension of the repeller may coincide with the dimension of the ambient manifold.

Example 1. Let $a_n = a/(n \log^2 n)$ for $n \geqslant 2$, where a is chosen so that $\sum_{n=2}^{\infty} a_n = 1$. Observe that, although a_n is summable, $\sum_{n=2}^{\infty} a_n^{\beta} = \infty$ for every $\beta < 1$. Let $b_n = \sum_{j \leqslant n} a_j$ for $n \geqslant 1$. Define $f\colon [0, 1] \to [0, 1]$ so that f maps each interval $[b_{n-1}, b_n)$ affinely onto $[0, 1)$. Then $Df \equiv 1/a_n$ on $[b_{n-1}, b_n)$. Fix any $p \geqslant 2$, let $H = (b_{p-1}, b_p)$, and Λ be the set of points $x \in [0, 1]$ whose orbit never enters H. Let q be any large integer, and $\Lambda_q \subset \Lambda$ be the set of points whose orbits remain forever in $[0, b_q] \setminus H$. It is well known, see e.g., ref. 7, p. 68 that the Hausdorff dimension d_q of Λ_q is the unique solution of

$$\sum_{n=2,\, n \neq p}^{q} a_n^{d_q} = 1$$

The fact that a_n^{β} is not summable for any $\beta < 1$ implies that $d_q \to 1$ when $q \to \infty$. So, $\mathrm{HD}(\Lambda) = 1$.

Remark 2. Suppose f is volume-expanding on $R_1 \cup \cdots \cup R_m$. Then the Lebesgue measure of n-cylinders decreases exponentially fast: there exists $c_0 > 0$ depending only on the map such that $\mathrm{Leb}(C(\alpha_1,\ldots, \alpha_n)) \leqslant e^{-c_0 n}$. Suppose there exists $\tilde{\beta} < 1$ such that

$$S = \sum_{n=1}^{\infty} \sum_{C \in \mathcal{Q}_n} \mathrm{Leb}(C)^{\tilde{\beta}} < \infty. \tag{5}$$

Then, for every $n \geqslant 1$,

$$\sum_{C \in \mathcal{Q}_n} \mathrm{Leb}(C) \leqslant \sum_{C \in \mathcal{Q}_n} \mathrm{Leb}(C)^{\tilde{\beta}}\, e^{-c_0(1-\tilde{\beta})\, n} \leqslant S\, e^{-c_0(1-\tilde{\beta})\, n}.$$

This means that (5) implies exponential decay as required in (A_2), with arbitrary $c_1 < c_0(1-\tilde{\beta})$.

2. INDUCING AN EXPANDING MAP

As a matter of fact, we are going to prove the conclusion of Theorem 1, under conditions more general than (A_1) and (A_2). Instead of requiring, as we did in (A_1), that the limit capacity of the boundary of each R_i be less than d, we just assume that there exist domains $R_i' \supset R_i$ with that property and such that the restriction $f \mid R_i$ may be extended diffeomorphically to the larger domain R_i'.

This weakening of our hypothesis is useful for the applications in ref. 6, where the existence of such extensions can be readily established (the R_i' may be taken with piecewise smooth boundaries), whereas the boundaries of the R_i are fractal, and we do not know whether their limit capacity is less than d (although this seems likely).

The precise condition is

(A_1') For each $1 \leqslant i \leqslant m$ there exists a domain $R_i' \supset R_i$ and an extension $f_i \colon R_i' \to W_i'$ of $f \mid R_i$ to a $C^{1+\varepsilon}$ diffeomorphism from R_i' onto a domain $W_i' \supset W_i$ that contains $R_1' \cup \cdots \cup R_m'$. Moreover, the limit capacity $c(\partial R_i')$ is less than d, and $H_i' = W_i' \backslash (R_1' \cup \cdots \cup R_m')$ has non-empty interior.

Observe that the R_i' need not be disjoint, and the extensions f_i need not coincide on the intersections. This means that (A_2) needs some reformulation.

We define the *extended n-cylinder* of f corresponding to an itinerary $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots)$ as follows. First, $C'(\alpha) = R_\alpha'$ for any α in $\{1, \ldots, m\}$. For each $n \geqslant 2$, the definition is by recurrence:

$$C'(\alpha_1, \alpha_2, \ldots, \alpha_n) = R_{\alpha_1}' \cap f_{\alpha_1}^{-1}(C'(\alpha_2, \ldots, \alpha_n)).$$

Clearly, the set of extended n-cylinders that intersect Λ forms a covering of the repeller, for each $n \geqslant 1$. Moreover, we denote

$$f_{\underline{\alpha}}^n(x) = f_{\alpha_n} \circ \cdots \circ f_{\alpha_1}(x) \qquad \text{and} \qquad f_{\underline{\alpha}}^{-n}(y) = f_{\alpha_1}^{-1} \circ \cdots \circ f_{\alpha_n}^{-1}(y).$$

The former defines a diffeomorphism from $C'(\alpha_1, \ldots, \alpha_n)$ onto W_{α_n}', and the latter is its inverse. We extend the notion of average least expansion, by

$$\phi_n'(\alpha_1, \ldots, \alpha_n) = \frac{1}{n} \sum_{j=1}^{n} \inf_{x \in C_j} \log \|Df_{\alpha_j}^{-1}(f_{\underline{\alpha}}^j(x))\|^{-1} \tag{6}$$

where $C_j = C'(\alpha_1, \ldots, \alpha_j)$. Finally, given $c > 0$, we consider the set $\mathcal{Q}_n'$ of extended n-cylinders that intersect Λ and for which $\phi_n'(\alpha_1, \ldots, \alpha_n) \leqslant c$. Condition ($A_2$) becomes

(A_2') There exist $c > 0$ and $c_1 > 0$ such that $\sum_{C \in \mathcal{Q}_n'} \operatorname{Leb}(C) \leqslant e^{-c_1 n}$ for every n sufficiently large.

It is clear that Theorem 1 is a particular case of the following result, corresponding to the case when $R_i' = R_i$ for all $1 \leqslant i \leqslant m$.

Theorem 2. Let $f\colon M \to M$ and Λ be as above, satisfying (A_1') and (A_2'). Then $\mathrm{HD}(\Lambda) < d$.

In the sequel we present the proof of Theorem 2. The first step is to define an *induced map* F whose domain E contains the set $\tilde{\Lambda}$ of points of Λ satisfying the hypothesis (A_2') for some n sufficiently large. F is piecewise smooth and expanding, with countably many domains of differentiability. Actually, there is some overlap between the different domains, and so F is really a multi-valued map. In Proposition 2.3 we prove that the Hausdorff dimension of the exceptional set $G = \Lambda \setminus \tilde{\Lambda}$ is strictly less than d.

Throughout, we suppose that the Riemannian metric has been rescaled, so that the volume and the diameter of every W_i' are less than 1.

Remark 3. Corresponding to Remark 1, it is sufficient to assume that every W_i' contains the union of all R_j' when j varies in the set $J(i)$ of indices for which R_j intersects W_i. Moreover, the iterates of every R_i' should eventually contain an R_k' such that $W_k' \setminus \bigcup_{j \in J(i)} R_j'$ has non-empty interior.

2.1. Hyperbolic Times

Given a local diffeomorphism f and a positive number ρ, Alves *et al.*[1,2] call *ρ-hyperbolic time* for a point $x \in M$ any integer $n \geqslant 0$ such that

$$\frac{1}{k} \sum_{j=n-k+1}^{n} \log \|Df^{-1}(f^j(x))\|^{-1} \geqslant \rho, \qquad \text{for every} \quad 1 \leqslant k \leqslant n.$$

The definition implies that $Df^k(f^{n-k}(x))$ is an expansion:

$$\|Df^{-k}(f^n(x))\| \leqslant \prod_{j=n-k+1}^{n} \|Df^{-1}(f^j(x))\| \leqslant e^{-\rho k}, \tag{7}$$

for every $1 \leqslant k \leqslant n$. We are going to use a slight variation of this notion to construct a piecewise expanding map induced by a map f as in Theorem 2, with control on the volume distortion.

Definition 1. Given $\rho > 0$ we say that $n \geqslant 0$ is a *ρ-hyperbolic time* for an extended n-cylinder $C'(\alpha_1,\ldots,\alpha_n)$ if

$$\frac{1}{k} \sum_{j=n-k+1}^{n} \inf_{x \in C_j} \log \|Df_{\alpha_j}^{-1}(f_{\alpha_j}^{j}(x))\|^{-1} > \rho, \qquad \text{for every} \quad 1 \leqslant k \leqslant n,$$

where $C_j = C'(\alpha_1,\ldots,\alpha_j)$. Moreover, $0 \leqslant h < n$ is a ρ-hyperbolic time for $C'(\alpha_1,\ldots,\alpha_n)$ if it is a ρ-hyperbolic time for $C'(\alpha_1,\ldots,\alpha_h)$.

The condition is void when $n=0$, and so zero is always a (trivial) hyperbolic time.

Lemma 2.1. Given any $c>\rho>0$ there exists $\theta>0$, depending only on c, ρ, and a uniform bound for $\|Df^{-1}\|$, such that if $\alpha_1,\dots,\alpha_n$ satisfy

$$\phi'_n(\alpha_1,\dots,\alpha_n)>c \tag{8}$$

then there are at least θn ρ-hyperbolic times $h_j\leqslant n$ for $C'(\alpha_1,\dots,\alpha_n)$.

Proof. The proof is analogous to Lemma 3.1 and Corollary 3.2 of ref. 2, with $a_j=\inf_{x\in C_j}\log\|Df_{\alpha_i}^{-1}(f^i_{\alpha_i}(x))\|^{-1}$. This sequence is bounded, because the maps f_α are C^1 diffeomorphisms on each compact set R'_α. ∎

We are going to use the following direct consequence: given any $q\geqslant 1$, if (8) holds for $n>(q/\theta)$ then $C'(\alpha_1,\dots,\alpha_n)$ has hyperbolic times $h\geqslant q$.

We fix c as in (A'_2) and $\rho=c/2$. We also fix $q\geqslant 1$ (here the value is arbitrary, but for applications one may want to choose it to be large), and let N be the smallest integer larger than q/θ. Let $\tilde{\Lambda}$ be the set of points $x\in\Lambda$ satisfying hypothesis (A'_2) for some $n\geqslant N$. By the previous considerations, every $x\in\tilde{\Lambda}$ is contained in some extended cylinder that has a $c/2$-hyperbolic time $h\geqslant q$. This puts us in a position to define the induced map F.

For every $h\geqslant q$, let $\mathscr{P}_h$ be the family of extended h-cylinders for which h is a hyperbolic time, and the smallest one after q: there is no other hyperbolic time in the interval $[q,h)$. For each $C'(\alpha_1,\dots,\alpha_h)$ in $\mathscr{P}_h$, we define

$$F\mid C'(\alpha_1,\dots,\alpha_h)=f^h_{\underline{\alpha}},\qquad \text{that is,}\quad F(x)=f_{\alpha_h}\circ\cdots\circ f_{\alpha_1}(x). \tag{9}$$

Thus, $F\mid C'(\alpha_1,\dots,\alpha_h)$ sends the cylinder diffeomorphically onto the domain W'_{α_h}. Moreover, according to the next lemma, this map is uniformly expanding.

Given any domain V in M, we denote by $d_V(x,y)$ the *distance between points* x, y *in* V, defined as the shortest length of a curve connecting x to y inside V. For notational simplicity, in the next lemma we write $d_j(\cdot,\cdot)$ to mean the distance in $f^j_{\underline{\alpha}}(C'(\alpha_1,\dots,\alpha_h))$, for each $0\leqslant j\leqslant h$.

Lemma 2.2. Given any $x_1, x_2\in C'(\alpha_1,\dots,\alpha_h)$ and $1\leqslant k\leqslant h$,

$$d_{h-k}(f^{h-k}_{\underline{\alpha}}(x_1), f^{h-k}_{\underline{\alpha}}(x_2))\leqslant e^{-ck/2}d_h(f^h_{\underline{\alpha}}(x_1), f^h_{\underline{\alpha}}(x_2)).$$

In particular, $d_{W'_i}(x_1,x_2)\leqslant e^{-cq/2}d_{W'_{\alpha_h}}(F(x_1),F(x_2))$ for all $i=1,\dots,m$.

Proof. Since h is a hyperbolic time

$$\sum_{j=h-k+1}^{h} \inf_{C_j} \log \|Df_{\alpha_j}^{-1}(f_{\underline{\alpha}}^{j}(x))\|^{-1} \geqslant \rho k = \frac{ck}{2}.$$

Just as in (7), this leads to

$$\|D(f_{\alpha_h} \circ \cdots \circ f_{\alpha_{h-k+1}})^{-1}(f_{\underline{\alpha}}^{n}(x))\| \leqslant \prod_{j=h-k+1}^{h} \|Df_{\alpha_j}^{-1}(f_{\underline{\alpha}}^{j}(x))\| \leqslant e^{-ck/2}$$

for every $x \in C'(\alpha_1, \ldots, \alpha_h)$. The first claim in the lemma follows: using the mean value theorem, backward iterates decrease the length of any curve by a factor $e^{-c/2}$.

The particular case $k = h$ reads $d_0(x_1, x_2) \leqslant e^{-ch/2} d_h(F(x_1), F(x_2))$. The term on the left is not smaller than $d_{W'_i}(x_1, x_2)$, because the cylinder is contained in W'_i, for every i. Moreover, $d_h(\cdot, \cdot)$ is the same as $d_{W'_{\alpha_h}}(\cdot, \cdot)$, because $f_{\underline{\alpha}}^{h}$ maps the cylinder onto W'_{α_h}. So the second part of the lemma also follows, recalling that $h \geqslant q$. ∎

Having been entirely precise in the formulation of this lemma, from now on we omit subscripts in the notation of the various distances, whenever the corresponding domain is clear from the context.

Let $\mathscr{P} = \bigcup_{h \geqslant q} \mathscr{P}_h$. We shall represent the elements of $\mathscr{P}$ as P'_j and denote $F_j = F \mid P'_j$, for $j \geqslant 1$. Let E be the union of all the elements of $\mathscr{P}$. By construction E contains $\tilde{\Lambda}$. We shall refer to E as the domain of F. Recall, however, that the extended cylinders may not be disjoint and so, in general, F is a multi-valued map.

Proposition 2.3. Let G be the complement of $\tilde{\Lambda}$ in Λ. Then $\mathrm{HD}(G) < d$.

This shows that the complement of $\tilde{\Lambda}$ is negligible as far as the proof of Theorem 2 is concerned. For convenience of presentation, we postpone the proof of this proposition to Section 4.2.

2.2. Bounded Volume Distortion

Here we prove that the Jacobian of F has a property of bounded distortion, cf. Proposition 2.5.

We call *inverse branch* of F any map of the form $F^{(-1)} = F_j^{-1}$. Recall that we denote $F_j = F \mid P'_j$, for each $P'_j = C'(\alpha_1, \ldots, \alpha_h)$ in $\mathscr{P}$. It sends W'_{α_h} diffeomorphically onto P'_j, which is contained in W'_i for any $1 \leqslant i \leqslant m$. For

$n \geqslant 2$, an inverse branch $F^{(-n)}$ of F^n is just a composition of n inverse branches of F.

Lemma 2.4. There exists a constant $C_0 > 0$ such that $\log |\det DF^{(-1)}|$ is a (C_0, ε)-Hölder map, for every inverse branch $F^{(-1)}$ of F.

Proof. By construction, $F^{(-1)} = f_{\underline{\alpha}}^{-h}$ for some $\underline{\alpha}$ and $h \geqslant q$. Let y_1, y_2 be points in the domain W'_{α_h} of $F^{(-1)}$, and $x_i = F^{(-1)}(y_i)$ for $i = 1, 2$. Thus,

$$\begin{aligned}\log |\det DF^{(-1)}(y_1)| &- \log |\det DF^{(-1)}(y_2)| \\ &= \sum_{j=0}^{h-1} \log |\det Df_{\alpha_{h-j}}^{-1}(f_{\underline{\alpha}}^{h-j}(x_1))| - \log |\det Df_{\alpha_{h-j}}^{-1}(f_{\underline{\alpha}}^{h-j}(x_2))|.\end{aligned}$$

By our smoothness requirement on f_i, in (A'_1), the Jacobian $\log |\det(Df_i^{-1})|$ is ε-Hölder continuous. Let $\tilde{C}_0$ be some Hölder constant for it. Moreover, by Lemma 2.2,

$$d(f_{\underline{\alpha}}^{h-j}(x_1), f_{\underline{\alpha}}^{h-j}(x_2)) \leqslant e^{-cj/2} d(y_1, y_2),$$

for every $1 \leqslant j \leqslant h$. So, $\log |\det DF^{(-1)}(y_1)| - \log |\det DF^{(-1)}(y_2)|$ is bounded by

$$\tilde{C}_0 \sum_{j=1}^{h} d(f_{\underline{\alpha}}^{h-j}(x_1), f_{\underline{\alpha}}^{h-j}(x_2))^{\varepsilon} \leqslant \tilde{C}_0 d(y_1, y_2)^{\varepsilon} \sum_{j=1}^{h} e^{-c\varepsilon j/2} \leqslant C_0 d(y_1, y_2)^{\varepsilon},$$

where $C_0 = \tilde{C}_0 \sum_{j=1}^{\infty} e^{-c\varepsilon j/2}$. ∎

Proposition 2.5 [Bounded Distortion]. There exists a constant $C_1 > 0$ such that

$$\frac{1}{C_1} \leqslant \frac{|\det DF^{(-n)}(y)|}{|\det DF^{(-n)}(z)|} \leqslant C_1$$

for every inverse branch $F^{(-n)}$ of F^n, any $n \geqslant 1$, and for every pair of points y, z in the domain of $F^{(-n)}$.

Proof. By definition, we may write $F^{(-n)} = h_n \circ \cdots \circ h_1$, where each h_i is an inverse branch of F. Then

$$\begin{aligned}\log \frac{|\det DF^{(-n)}(y)|}{|\det DF^{(-n)}(z)|} = \sum_{j=1}^{n} &(\log |\det Dh_j(h_{j-1} \circ \cdots \circ h_1)(y)| \\ &- \log |\det Dh_j(h_{j-1} \circ \cdots \circ h_1)(z)|).\end{aligned}$$

By the previous lemma, each function $\log|\det Dh_j|$ is (C_0, ε)-Hölder. By Lemma 2.2, every h_i is an $e^{-cq/2}$-contraction, Then

$$\log\frac{|\det DF^{(-n)}(y)|}{|\det DF^{(-n)}(z)|} \leqslant \sum_{j=1}^{n} C_0\, d((h_{j-1}\circ\cdots\circ h_1)(y), (h_{j-1}\circ\cdots\circ h_1)(z))^{\varepsilon}$$

$$\leqslant \sum_{j=0}^{n-1} C_0\, e^{-cqj\varepsilon/2} d(y,z)^{\varepsilon} \leqslant C_0\, d(y,z)^{\varepsilon} \sum_{j=0}^{\infty} e^{-cqj\varepsilon/2}.$$

Take $C_1 = \exp(C_0 \rho_0^{\varepsilon} \sum_{j=0}^{\infty} e^{-cqj\varepsilon/2})$, where ρ_0 is some uniform upper bound for the diameter of the domains W_i' of inverse branches. It follows that

$$\frac{|\det DF^{(-n)}(y)|}{|\det DF^{(-n)}(z)|} \leqslant C_1.$$

The other inequality is obtained reversing the roles of y and z. ▮

Corollary 2.6. Let $C_2 = C_1^2$. Then, given $n \geqslant 1$ and any inverse branch $F^{(-n)}$ of F^n, we have

$$\frac{1}{C_2}\frac{\mathrm{Leb}(A)}{\mathrm{Leb}(B)} \leqslant \frac{\mathrm{Leb}(F^{(-n)}(A))}{\mathrm{Leb}(F^{(-n)}(B))} \leqslant C_2 \frac{\mathrm{Leb}(A)}{\mathrm{Leb}(B)}.$$

for any measurable subsets A and B of the domain of $F^{(-n)}$.

Proof. Fix some point z in the domain of $F^{(-n)}$. Then

$$\frac{\mathrm{Leb}(F^{(-n)}(A))}{\mathrm{Leb}(F^{(-n)}(B))} = \frac{\int_A |\det DF^{(-n)}(x)|\,dx}{\int_B |\det DF^{(-n)}(y)|\,dy}$$

$$= \int_A \frac{|\det DF^{(-n)}(x)|}{|\det DF^{(-n)}(z)|}\,dx \Big/ \int_B \frac{|\det DF^{(-n)}(y)|}{|\det DF^{(-n)}(z)|}\,dy.$$

Hence, by the previous proposition,

$$\frac{1}{C_1^2}\frac{\mathrm{Leb}(A)}{\mathrm{Leb}(B)} \leqslant \frac{\mathrm{Leb}(F^{(-n)}(A))}{\mathrm{Leb}(F^{(-n)}(B))} \leqslant C_1^2 \frac{\mathrm{Leb}(A)}{\mathrm{Leb}(B)}.$$

This proves the corollary. ▮

3. DYNAMICAL DIMENSION

Notions of dimension of a set with respect to a dynamical system have been considered by several authors, see e.g., refs. 3 and 8. In very brief terms, one mimics the definition of Hausdorff dimension, with diameter replaced by volume. Moreover, one considers only certain covering sets, that are dynamically generated. Thus, the dynamical dimension is not a purely geometric invariant, it may depend also on the underlying dynamical system.

Here we use a variation of this notion suitable for multi-valued maps. Let M be a d-dimensional Riemannian manifold, and $\mathscr{P} = \{P'_j : j \geqslant 1\}$ be a countable family of sub-domains of M. Suppose, for each $P'_j \in \mathscr{P}$ a continuous map $F_j : P'_j \to M$ is given that sends P'_j bijectively onto some domain that contains $V' = \bigcup_{i \geqslant 1} P'_i$. Given $k \geqslant 1$, the *k-cylinder* of F associated to a sequence $(j_1,..., j_k) \in \mathbb{N}^k$ is the image $\tilde{C}_n(j_1,..., j_k)$ of the inverse branch

$$F^{(-k)} = F_{j_1}^{-1} \circ \cdots \circ F_{j_k}^{-1}.$$

Given any $\beta > 0$ and $\Lambda \subset V'$, we define the β-dimensional dynamical measure of Λ with respect to F to be

$$m_\beta(\Lambda, F) = \lim_{\delta \to 0} \inf_{|\mathscr{U}| < \delta} \sum_{U \in \mathscr{U}} \mathrm{Leb}(U)^\beta, \tag{10}$$

where Leb is the Riemannian volume in M, and the infimum is taken over all coverings $\mathscr{U}$ of Λ by cylinders with diameter less than δ. If there is no such covering then $m_\beta(\Lambda, F)$ is infinite, by convention.

In general, it is easy to see that there exists $0 \leqslant \bar{\beta} \leqslant 1$ such that

$$m_\beta(\Lambda, F) = \infty \quad \text{for} \quad \beta < \bar{\beta} \qquad \text{and} \qquad m_\beta(\Lambda, F) = 0 \quad \text{for} \quad \beta > \bar{\beta}.$$

We define the *dynamical dimension of* Λ with respect to F to be

$$DD_F(\Lambda) = d \sup\{\beta : m_\beta(\Lambda, F) = \infty\} = d \inf\{\beta : m_\beta(\Lambda, F) = 0\}.$$

Recall that d is the dimension of the ambient manifold M.

The main result in this section, and a step towards proving Theorem 2, is the following

Proposition 3.1. Let $\tilde{\Lambda}$ be the subset of the repeller Λ defined in Section 2. There is a constant $\beta_0 < 1$ such that

$$DD_F(\tilde{\Lambda}) \leqslant d\beta_0 < d.$$

The assumption about the limit capacity of the boundaries in (A_1), (A'_1) will not be used at all until Section 4. In particular, Proposition 3.1 remains true without it. However, for the arguments in the next section we need the coverings of $\tilde{\Lambda}$ that we are going to construct for proving the proposition to have limit capacity of the boundaries smaller than d. That is the reason why we assume, already at this point, that the R'_i have such a property.

Note also that, since f_i is a C^1 diffeomorphism, both it and its inverse are Lipschitz continuous. It follows that they preserve geometric invariants such as the limit capacity and the Hausdorff dimension; see ref. 7, Chapter 4. In particular, the boundary of W'_i has the same limit capacity as $\partial R'_i$.

We split the proof of Proposition 3.1 into several short lemmas. In what follows P'_j is a generic element of $\mathscr{P}$. That is P'_j is an extended h-cylinder $C'(\alpha_1,\ldots,\alpha_h)$, with h being the first hyperbolic time after some fixed $q \geqslant 1$. Recall that we called N the smallest integer large than (q/θ).

The first lemma states that if the inducing time h is large then P'_j is in the exceptional class $\mathscr{Q}'_h$ of hypothesis (A'_2).

Lemma 3.2. If $h > 2N$ then $\phi'_h(\alpha_1,\ldots,\alpha_h) \leqslant c$.

Proof. Otherwise there would be at least $h\theta > 2N\theta$ hyperbolic times less or equal than h. Since $2N\theta > 2q$ there would be at least $2q+1 \geqslant q+2$ such hyperbolic times. Then, at least one of them would be in $[q, h)$, contradicting the definition of h. ∎

Lemma 3.3. There exist constants $C_3 > 0$ and $\eta < 1$ such that

$$\mathrm{Leb}(C'(\alpha_1,\ldots,\alpha_n)) \geqslant C_3\eta^n$$

for any $n \geqslant 1$ and any extended n-cylinder $C'(\alpha_1,\ldots,\alpha_n)$.

Proof. By construction, $f^n_{\underline{\alpha}}$ maps the cylinder diffeomorphically onto W'_{α_n}. Therefore,

$$\mathrm{Leb}(C'(\alpha_1,\ldots,\alpha_n)) \geqslant \frac{\mathrm{Leb}(W'_{\alpha_n})}{\sup|\det Df^n_{\underline{\alpha}}|}.$$

Let C_3 be a lower bound for the Lebesgue measure of the domains W'_i. Also, since f is a diffeomorphism on each compact set R'_i, its Jacobian is bounded above by some constant $1/\eta$. It follows that the volume of the n-cylinder is at least $C_3\eta^n$. ∎

Corollary 3.4. There exist $\beta_1 < 1$ such that, for any $\beta_1 \leqslant \beta < 1$,

$$\sum_{j=1}^{\infty} \mathrm{Leb}(P'_j)^{\beta} < \infty.$$

Proof. The total number of h-cylinders with $h \leqslant 2N$ is finite and, in fact, less than $2Nm^{2N}$. Therefore, the sum over all the P'_j with $h \leqslant 2N$

$$\sum_{h \leqslant 2N} \mathrm{Leb}(P'_j)^{\beta}$$

is always finite, for any β. So, to prove the statement we only have to show that the sum over all the P'_j with $h > 2N$ is also finite. Now, by Lemma 3.3,

$$\sum_{h > 2N} \mathrm{Leb}(P'_j)^{\beta} = \sum_{h > 2N} \mathrm{Leb}(P'_j)\,\mathrm{Leb}(P'_j)^{\beta-1} \leqslant \sum_{h > 2N} \mathrm{Leb}(P'_j)(C_3\eta^h)^{\beta-1}.$$

Then, using Lemma 3.2 and (A'_2),

$$\sum_{h > 2N} \mathrm{Leb}(P'_j)^{\beta} \leqslant \sum_{h > 2N} (C_3\eta^h)^{\beta-1} \sum_{C \in \mathscr{Q}'_h} \mathrm{Leb}(C) \leqslant \sum_{h > 2N} (C_3\eta^h)^{\beta-1}\, e^{-hc_1}.$$

Assuming that β is close enough to 1, the ration of this geometric series is smaller than 1, and so the series converges. Indeed, it is enough to suppose that $\beta \in [\beta_1, 1)$ for some fixed $\beta_1 < 1$ close enough to 1 so that $\eta^{\beta_1 - 1}e^{-c_1}$ is smaller than 1. ∎

For each $j \geqslant 1$, let $F^{(-1)}\colon W'_{\alpha_h} \to P'_j$ be the inverse branch associated to P'_j, that is, the inverse of F_j. For each $j \geqslant 1$, we consider the following subsets of P'_j:

$$P_j = F^{(-1)}(W_{\alpha_h}) \qquad \text{and} \qquad B_j = F^{(-1)}(H_{\alpha_h}).$$

See Fig. 4.

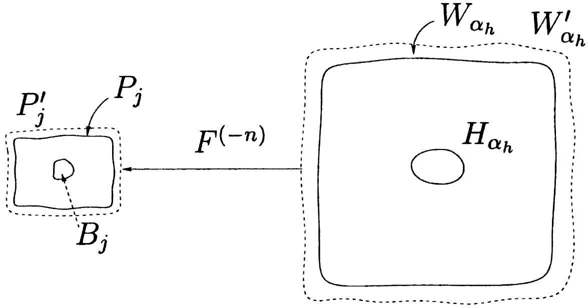

Fig. 4. Definition of P_j and B_j.

Recall that $H_i = W_i \setminus (R_1 \cup \cdots \cup R_m)$. So, the definitions imply that $P_j \setminus B_j = F^{(-1)}(R_1 \cup \cdots \cup R_m)$. We consider the family of subsets

$$\mathscr{U}_1 = \{(P_j \setminus B_j)^\circ = F^{(-1)}((R_1 \cup \cdots \cup R_m)^\circ) \colon j \geqslant 1\}, \tag{11}$$

where A° is the interior of a set A in the ambient manifold. The following fact is an immediate consequence of Corollary 3.4:

Corollary 3.5. For any $\beta \in [\beta_1, 1)$,

$$\sum_{U_1 \in \mathscr{U}_1} \mathrm{Leb}(U_1)^\beta = \sum_{j=1}^{\infty} \mathrm{Leb}((P_j \setminus B_j)^\circ)^\beta < \infty.$$

The next lemma shows that P_j is just the (regular) cylinder associated to the extended cylinder P'_j. In particular, the P_j have two-by-two disjoint interiors.

Lemma 3.6. If $P'_j = C'(\alpha_1,..., \alpha_h)$ then $P_j = C(\alpha_1,..., \alpha_h)$. In particular, $P_j^\circ \cap P_k^\circ = \emptyset$ if $j \neq k$.

Proof. Recall that $F^{(-1)} \colon W'_{\alpha_h} \to R'_1$ is defined by $F^{(-1)} = f_{\alpha_1}^{-1} \circ \cdots \circ f_{\alpha_h}^{-1}$. We have $f_{\alpha_h}^{-1}(W_{\alpha_h}) = R_{\alpha_h} \subset W_{\alpha_h}$, because f_{α_h} is a bijective extension of $f \mid R_{\alpha_h}$, and the latter sends R_{α_h} onto W_{α_h}. Then

$$f_{\alpha_{h-1}}^{-1} \circ f_{\alpha_h}^{-1}(W_{\alpha_h}) = f_{\alpha_{h-1}}^{-1}(R_{\alpha_h}) = R_{\alpha_{h-1}} \cap f^{-1}(R_{\alpha_h}) = C(\alpha_{h-1}, \alpha_h).$$

After h analogous steps we find that $f_{\alpha_1}^{-1} \circ \cdots \circ f_{\alpha_h}^{-1}(W_{\alpha_h}) = C(\alpha_1,..., \alpha_h)$, just as claimed in the first part of the lemma.

Note that two cylinders either have disjoint interiors or one of them is contained in the other. So, to prove the second part we only have to check that given any other $P_k = C(\beta_1,..., \beta_l)$ different from P_j, neither $P_j \subset P_k$ nor $P_k \subset P_j$. Indeed, $P_j \subset P_k$ would imply $h \geqslant l$ and $\alpha_i = \beta_i$ for $1 \leqslant i \leqslant l$. Then either $h = l$, in which case $P_j = P_k$, or else $h > l$, which contradicting the choice of h as the first hyperbolic time. The case $P_k \subset P_j$ is entirely analogous. ∎

Next, we are going to construct a sequence $\mathscr{U}_n$ of families of subsets such that

$$\sum_{U_n \in \mathscr{U}_n} \mathrm{Leb}(U_n)^\beta \tag{12}$$

is non-increasing, for $\beta < 1$ close enough to 1. The first step was (11). The general one is by recurrence: $\mathcal{U}_{n+1}$ is obtained replacing each $U_n \in \mathcal{U}_n$ by the family of subsets that one obtains by pull-back of $\mathcal{U}_1$ to U_n under F^n. In detail, this goes as follows.

Suppose a family $\mathcal{U}_n$ has been constructed, such that every element U_n of $\mathcal{U}_n$ is contained in an n-cylinder $\tilde{C}(j_1,..., j_n)$ of F, in fact

$$U_n = F^{(-n)}((R_1 \cup \cdots \cup R_m)^\circ)$$

where $F^{(-n)}$ is the inverse branch of F^n corresponding to that cylinder. For each $j \geqslant 1$, let $F^{(-1)}$ be the inverse of F_j and

$$U_{n+1,j} = F^{(-n)}((P_j \setminus B_j)^\circ) = F^{(-n)} \circ F^{(-1)}((R_1 \cup \cdots \cup R_m)^\circ).$$

This is well-defined because $P_j \subset R_{\alpha_1}$, whereas $F^{(-n)}$ is defined on some domain $W'_i \supset R_{\alpha_1}$. Moreover, $U_{n,j}$ is contained in the $(n+1)$-cylinder $\tilde{C}(j_1,..., j_n, j)$. We take $\mathcal{U}_{n+1}$ to be the family of all sets $U_{n+1,j}$ obtained in this way, for all $U_n \in \mathcal{U}_n$.

In what follows, $F^{(-n-1)} = F^{(-n)} \circ F^{(-1)}$. Let us introduce the numerical sequences

- $x_j = \dfrac{\mathrm{Leb}(F^{(-n)}((P_j \setminus B_j)^\circ))}{\mathrm{Leb}(U_n)} = \dfrac{\mathrm{Leb}(F^{(-n-1)}((R_1 \cup \cdots \cup R_m)^\circ))}{\mathrm{Leb}(U_n)}$
- $y_j = \dfrac{\mathrm{Leb}(F^{(-n)}(B_j^\circ))}{\mathrm{Leb}(U_n)} = \dfrac{\mathrm{Leb}(F^{(-n-1)}(H_{\alpha_h}^\circ))}{\mathrm{Leb}(U_n)}$
- $z_j = \dfrac{\mathrm{Leb}(F^{(-n)}(P_j^\circ))}{\mathrm{Leb}(U_n)} = \dfrac{\mathrm{Leb}(F^{(-n-1)}(W_{\alpha_h}^\circ))}{\mathrm{Leb}(U_n)} \geqslant x_j + y_j.$

Our immediate goal is to prove that $\sum_{j=1}^{\infty} x_j^\beta \leqslant 1$ if β is close enough to 1. We are going to do that with the help of the following elementary fact, that we borrow from ref. 4, Lemma 3.1.

Lemma 3.7. Given $a > 0$, $A > 0$, and $\alpha < 1$ there exists $\bar{\beta} < 1$ such that if $(x_j)_j$ and $(y_j)_j$ are any sequences of positive real numbers such that

1. $\sum_j (x_j + y_j) \leqslant 1$,
2. $x_j \leqslant a y_j$ for every j,
3. $\sum_j (x_j + y_j)^\alpha \leqslant A$,

then $\sum_j x_j^\beta \leqslant 1$ for every $\beta \in [\bar{\beta}, 1)$.

That our sequences do satisfy the hypotheses of this lemma, is guaranteed by the next couple of results.

Lemma 3.8. There exists $a > 0$ such that $x_j \leqslant a y_j$ for every j.

Proof. This is a direct consequence of Corollary 2.6:

$$\frac{x_j}{y_j} = \frac{\mathrm{Leb}(F^{(-n-1)}((R_1 \cup \cdots \cup R_m)^\circ))}{\mathrm{Leb}(F^{(-n-1)}(H^\circ_{\alpha_h}))} \leqslant C_2 \frac{\mathrm{Leb}(R_1 \cup \cdots \cup R_m)}{\mathrm{Leb}(H^\circ_{\alpha_h})}$$

and so it suffices to take $a = (C_2/C_4)\,\mathrm{Leb}(R_1 \cup \cdots \cup R_m)$ where $C_4 > 0$ is a lower bound for the Lebesgue measure of H°_i, with $1 \leqslant i \leqslant m$. ∎

Lemma 3.9. Let $\beta_1 < 1$ be as in Corollary 3.4. There exists $A > 0$ such that

$$\sum_j z_j \leqslant 1 \quad \text{and} \quad \sum_j z_j^{\beta_1} \leqslant A.$$

Proof. The first claim follow directly from Lemma 3.6:

$$\sum_j z_j = \sum_j \frac{\mathrm{Leb}(F^{(-n)}(P^\circ_j))}{\mathrm{Leb}(U_n)} \leqslant \frac{\mathrm{Leb}(F^{(-n)}(\bigcup_j P^\circ_j))}{\mathrm{Leb}(F^{(-n)}((R_1 \cup \cdots \cup R_m)^\circ))} \leqslant 1$$

because the P_j have two-by-two disjoint interiors, and each of them is contained in some R_i.

To prove the second part of the lemma, we begin by using Corollary 2.6:

$$z_j = \frac{\mathrm{Leb}(F^{(-n)}(P^\circ_j))}{\mathrm{Leb}(F^{(-n)}((R_1 \cup \cdots \cup R_m)^\circ))} \leqslant C_2 \frac{\mathrm{Leb}(P_j)}{\mathrm{Leb}((R_1 \cup \cdots \cup R_m)^\circ)}.$$

Thus, $z_j \leqslant C_5\,\mathrm{Leb}(P_j)$, where C_5 stands for $C_2/\mathrm{Leb}((R_1 \cup \cdots \cup R_m)^\circ)$. Then, by Corollary 3.4,

$$\sum_j z_j^\beta \leqslant \sum_j C_5^\beta\,\mathrm{Leb}(P_j)^\beta \leqslant \sum_j C_5^\beta\,\mathrm{Leb}(P'_j)^\beta$$

for every $\beta \in [\beta_1, 1)$. It suffices to take for A the value of this last sum when $\beta = \beta_1$ (the sum decreases when β increases). ∎

Now it is easy to prove that the sequence in (12) is non-increasing, as we announced.

Corollary 3.10. There exists $\beta_2 < 1$ such that, for every $n \geqslant 1$,

$$\sum_{U_{n+1} \in \mathcal{U}_{n+1}} \mathrm{Leb}(U_{n+1})^\beta \leqslant \sum_{U_n \in \mathcal{U}_n} \mathrm{Leb}(U_n)^\beta$$

Proof. Let a and A be as in Lemmas 3.8 and 3.9, and $\alpha = \beta_1$. Take $\beta_2 = \bar{\beta}$, as given by Lemma 3.7. For each $U_n \in \mathcal{U}_n$, let $U_{n+1,j}$ be the elements of $\mathcal{U}_{n+1}$ associated to it, as constructed above. Recall that $x_j = \mathrm{Leb}(U_{n+1,j})/\mathrm{Leb}(U_n)$. So, according to Lemma 3.7,

$$\sum_j \mathrm{Leb}(U_{n+1,j})^\beta = \sum_j x_j^\beta \,\mathrm{Leb}(U_n)^\beta \leqslant \mathrm{Leb}(U_n)^\beta$$

It follows that

$$\sum_{U_{n+1} \in \mathcal{U}_{n+1}} \mathrm{Leb}(U_{n+1})^\beta = \sum_{U_n \in \mathcal{U}_n} \sum_j \mathrm{Leb}(U_{n+1,j})^\beta \leqslant \sum_{U_n \in \mathcal{U}_n} \mathrm{Leb}(U_n)^\beta.$$

as claimed. ∎

Now we are ready to complete the proof of Proposition 3.1.

Proof. Let $\mathcal{U}_n$ be as before. For each $n \geqslant 1$ and $U_n = F^{(-n)}((R_1 \cup \cdots \cup R_m)^\circ)$ in $\mathcal{U}_n$, let U'_n be the image of the inverse branch $F^{(-n)}$ (image of the whole domain of the map). Let $\mathcal{U}'_n$ be the family of all U'_n obtained in this way, for each fixed n. Every $\mathcal{U}'_n$ is a covering of $\tilde{\Lambda}$ by n-cylinders of F. Since the inverse branches of F are uniformly contracting, by Lemma 2.2, the maximum diameter of all $U'_n \in \mathcal{U}'_n$ goes to zero when n goes to ∞. Therefore,

$$m_\beta(\tilde{\Lambda}, F) \leqslant \liminf_{n \to \infty} \sum_{U'_n \in \mathcal{U}'_n} \mathrm{Leb}(U'_n)^\beta$$

for every β. We choose $\beta_0 = \beta_2$, and claim that for every $\beta \in [\beta_0, 1)$ there exists $A_\beta > 0$ such that

$$\sum_{U'_n \in \mathcal{U}'_n} \mathrm{Leb}(U'_n)^\beta \leqslant A_\beta \quad \text{for all} \quad n \geqslant 1. \tag{13}$$

(One may take $A_\beta = A_{\beta_0}$ for all β, because the left hand side decreases when the exponent increases.) This implies that $m_\beta(\tilde{\Lambda}, F) \leqslant A_\beta < \infty$ for $\beta \geqslant \beta_0$. So, $DD_F(\tilde{\Lambda}) \leqslant \beta_0 d$, as we stated. Thus, we have reduced the proposition to proving this claim.

On its turn, (13) is a fairly easy consequence of Corollary 3.10 and bounded distortion. Indeed, Corollary 2.6 gives

$$\frac{\mathrm{Leb}(U'_n)}{\mathrm{Leb}(U_n)} = \frac{\mathrm{Leb}(F^{(-n)}(W'_i))}{\mathrm{Leb}(F^{(-n)}((R_1 \cup \cdots \cup R_m)^\circ))} \leqslant C_2 \frac{\mathrm{Leb}(W'_i)}{\mathrm{Leb}((R_1 \cup \cdots \cup R_m)^\circ)}$$

for each $U_n \in \mathcal{U}_n$ and the corresponding $U'_n \in \mathcal{U}'_n$, with W'_i being the domain of the inverse branch $F^{(-n)}$. The expression on the right hand side is, clearly, bounded by some constant C_6. It follows that

$$\sum_{U'_n \in \mathcal{U}'_n} \mathrm{Leb}(U'_n)^\beta \leqslant C_6^\beta \sum_{U_n \in \mathcal{U}_n} \mathrm{Leb}(U_n)^\beta.$$

According to Corollary 3.10, this sequence on the right is non-increasing. In particular, it is bounded by some constant $A_\beta > 0$ independent of n. This proves the claim (13). The proof of Proposition 3.1 is complete. ∎

4. HAUSDORFF DIMENSION

Our purpose is to prove Theorem 2. We are going to use the coverings $\mathcal{U}_n$ that were constructed in Section 3. Roughly speaking, we replace each element of $\mathcal{U}_n$ by an appropriate covering of it with balls. Because the volume of a ball is given by a power of its diameter, this allows us to transform volume estimates such as (13) into diameter estimates suitable for computing the Hausdorff dimension. In this way we prove that the Hausdorff dimension of $\tilde{\Lambda}$ is less than d. A similar argument also gives that the same is true for the exceptional set G, as stated in Proposition 2.3.

4.1. From Volume to Diameter

We begin by describing a general construction of efficient coverings by balls for sub-domains whose boundary is not too fractal, in the sense that the limit capacity of the boundary is less than the ambient dimension. Then every iterate of the domain under a local diffeomorphism may be covered by balls of appropriate size, such that the total volume of these balls is the same as the volume of the iterate, up to a factor that depends only on the dimension.

Proposition 4.1. Let R be a domain in some manifold such that limit capacity of the boundary ∂R is less than $d = \dim M$. Let $g: M \to M$ be a local diffeomorphism. Then there exists $\rho > 0$ such that for every $n \geqslant 1$

there exists a covering $\mathscr{B}$ of $g^n(R)$ by open balls of radius at least ρ^n such that

$$\sum_{B \in \mathscr{B}} \mathrm{Leb}(B) \leqslant C_0 \, \mathrm{Leb}(g^n(R)),$$

where C_0 depends only on R and the dimension d.

Proof. Let $K > 1$ be an upper bound for the norm and the determinant of both Dg and its inverse Dg^{-1}. We are going to define the covering $\mathscr{B}$ in two steps: first we include a family of balls that cover a ρ^n neighbourhood of the boundary; then we add another family, that covers the complement of this neighbourhood. For the time being, ρ is arbitrary: we make our choice at the end of the proof.

Covering a Neighbourhood of the Boundary. Let $d_1 =$ limit capacity of ∂R and d_2 be fixed in the open interval (d_1, d). By definition of limit capacity, there exists $C_1 > 0$ depending only on the domain R, and for every $\varepsilon > 0$ there exists a covering of ∂R by not more than $C_1 \varepsilon^{-d_2}$ balls of radius ε. In particular, we may cover ∂R with not more than $C_1 (K^n \rho^n)^{-d_2}$ balls $B(x_i, K^n \rho^n)$ of radius $K^n \rho^n$. The images $g^n(B(x_i, 2K^n \rho^n))$ are contained in the balls $B(g^n(x_i), 2K^{2n} \rho^n)$ of radius $2K^{2n} \rho^n$. By definition, these last balls are in our covering $\mathscr{B}$.

Let us observe that the total volume of the $B(g^n(x_i), 2K^{2n} \rho^n)$ is bounded by

$$C_1 (K^n \rho^n)^{-d_2} \, C_2 \, (2K^{2n} \rho^n)^d \tag{14}$$

where C_2 depends only on the dimension d. Moreover, we claim that these balls cover the ρ^n-neighbourhood V_n of the boundary of $g^n(R)$. This can be checked as follows. Since g is an open map, $\partial(g^n(R))$ is contained in $g^n(\partial R)$. Then, for every $y \in g^n(R)$ such that the distance from y to $\partial g^n(R))$ is less than ρ^n, there exists $x \in \partial R$ such that $g^n(x) \in \partial g^n(R)$ and $d(y, g^n(x)) \leqslant \rho^n$. See Fig. 5. Then, $d(g^{-n}(y), x) \leqslant K^n \rho^n$. Moreover, x belongs to some $B(x_i, K^n \rho^n)$, because these balls cover ∂R. Therefore, $g^{-n}(y) \in B(x_i, 2K^n \rho^n)$. Consequently, y belongs to $B(g^n(x_i), 2K^{2n} \rho^n)$. This proves the claim.

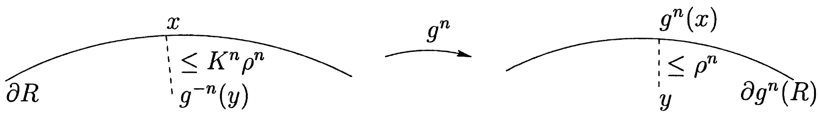

Fig. 5. Covering a neighbourhood of the boundary.

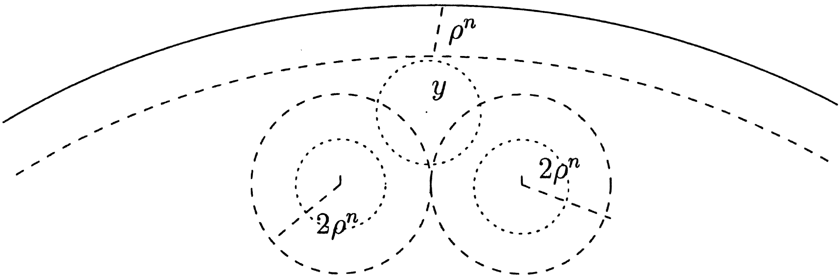

Fig. 6. Covering the interior of cylinders.

Covering the Interior. Consider a maximal family $B(y_j, \rho^n)$ of balls of radius ρ^n two-by-two disjoint and contained in $g^n(R)$. We add the balls $B(y_j, 2\rho^n)$ with the same centers and twice the radius to our covering $\mathscr{B}$. At this point the definition of $\mathscr{B}$ is complete. Let us observe that the $B(y_j, 2\rho^n)$ cover the $g^n(R) \setminus V_n$. Indeed, suppose there was $y \in g^n(R) \setminus V_n$ that is not contained Then, the ball of radius ρ^n around y would be disjoint from every $B(y_j, \rho^n)$, and so it could be added to the maximal family, which is a contradiction. This proves that the whole family $\mathscr{B}$ does cover the domain $g^n(R)$. See Fig. 6.

Finally, the total volume of these $B(y_j, 2\rho^n)$ is bounded by

$$2^d \sum_j \operatorname{Leb}(B(y_j, \rho^n)) \leqslant 2^d \operatorname{Leb}(g^n(R)). \tag{15}$$

So, adding (14) and (15), the total volume of the elements in the family $\mathscr{B}$ is bounded by

$$C_1 C_2 2^d (K^{2d-d_2} \rho^{d-d_2})^n + 2^d \operatorname{Leb}(g^n(R)).$$

To conclude the proof we only have to show that the first term is less than $C_3 \operatorname{Leb}(g^n(R))$ for some constant $C_3 > 0$. We choose

$$\rho = K^{-(2d-d_2+1)/(d-d_2)}.$$

Then,

$$(K^{2d-d_2} \rho^{d-d_2})^n = K^{-n} \leqslant \frac{\operatorname{Leb}(g^n(R))}{\operatorname{Leb}(R)} \qquad \text{for every} \quad n \geqslant 1.$$

So, we may take $C_3 = C_1 C_2 2^d / \operatorname{Leb}(R)$, and then $C_0 = C_3 + 2^d$. ∎

Remark 4. The arguments remain valid if one replaces g^n by a composition $g_n \circ \cdots \circ g_1$ of different local diffeomorphisms admitting a uniform bound for the norm and determinant of Dg_i and Dg_i^{-1}.

4.2. Proof of Theorem 2

We begin by proving Proposition 2.3: the exceptional subset G of Λ formed by those points that are not in $\tilde{\Lambda}$ has Hausdorff dimension less than d.

Proof. Recall that, $\tilde{\Lambda}$ was defined as the set of points of Λ that satisfy the condition in (A_2') for some $n > N$. Therefore, its complement is

$$G = \bigcap_{n \geqslant N} \bigcup_{C \in \mathcal{Q}_n'} C.$$

Let $C = C'(\alpha_1, \ldots, \alpha_n)$ be in $\mathcal{Q}_n'$. Then $C = f_{\underline{\alpha}}^{-n}(W_{\alpha_n}')$. Since every f_α is a diffeomorphism in its domain, and we assumed that the boundaries of the W_i' have limit capacity less than d, we may apply Proposition 4.1 (in the version of Remark 4): there exist $\rho > 0$ and C_0, depending only on the map and the domain W_i', such that C admits a covering $\mathcal{B}(C)$ by balls of radius at least ρ^n such that

$$\sum_{B \in \mathcal{B}(C)} \operatorname{Leb}(B) \leqslant C_0 \operatorname{Leb}(C).$$

Let $\mathcal{B}_n$ be the union of the $\mathcal{B}(C)$ over all $C \in \mathcal{Q}_n'$. Observe that every $\mathcal{B}_n$ covers G. By (A_2'),

$$\sum_{B \in \mathcal{B}_n} \operatorname{Leb}(B) = \sum_{C \in \mathcal{Q}_n'} \sum_{B \in \mathcal{B}(C)} \operatorname{Leb}(B) \leqslant C_0 \sum_{C \in \mathcal{Q}_n'} \operatorname{Leb}(C) \leqslant C_0 e^{-c_1 n}.$$

There exists a constant C_d that depends only on the dimension d such that $\operatorname{Leb}(B) \geqslant C_d \operatorname{diam}(B)^d$ for every ball. In particular, $\operatorname{Leb}(B) \geqslant C_d 2^d \rho^{nd}$ for all $B \in \mathcal{B}_n$, because their radii are not smaller than ρ^n. Then, given any $\gamma < 1$, we have

$$\begin{aligned}\sum_{B \in \mathcal{B}_n} \operatorname{Leb}(B)^\gamma &\leqslant \sum_{B \in \mathcal{B}_n} \operatorname{Leb}(B) \sup(\operatorname{Leb}(B)^{\gamma-1}) \\ &\leqslant C_0 e^{-c_1 n}\, C_d^{\gamma-1} 2^{d(\gamma-1)} \rho^{nd(\gamma-1)}.\end{aligned}$$

It follows that

$$\sum_{B \in \mathcal{B}_n} \operatorname{diam}(B)^{\gamma d} \leqslant C_0 C_d^{-1} 2^{d(\gamma-1)} e^{-c_1 n} \rho^{nd(\gamma-1)}.$$

Fix γ_1 close enough to 1 so that the the ratio $e^{-c_1}\rho^{d(\gamma_1-1)}$ is less than 1. Then, for any $\gamma \in [\gamma_1, 1)$,

$$\sum_{B \in \mathscr{B}_n} \operatorname{diam}(B)^{\gamma d} \to 0 \qquad \text{when} \quad n \to \infty.$$

On the other hand, the diameters of these coverings $\mathscr{B}_n$ go to zero when $n \to \infty$. This proves that the Hausdorff (γd)-measure $m_{\gamma d}(G)$ is zero for every $\gamma \geqslant \gamma_1$. So, the Hausdorff dimension of G is at most $\gamma_1 d < d$. ∎

Next we show that the Hausdorff dimension of $\tilde{\Lambda}$ is also less than d. In fact, our arguments show that $DD_F(\tilde{\Lambda}) < d$ implies $HD(\tilde{\Lambda}) < d$, whenever the following two additional conditions are satisfied: cylinders have boundaries with limit capacity less than d, and the volumes of n-cylinders decrease exponentially with n.

Proposition 4.2. We have $\mathrm{HD}(\tilde{\Lambda}) < d$.

Proof. For proving Proposition 3.1 we found $\beta_2 < 1$ and a sequence of coverings $\mathscr{U}_n$ of $\tilde{\Lambda}$ by n-cylinders of F, such that

$$S_0 = \sup_{n \geqslant 1} \sum_{U_n \in \mathscr{U}_n} \operatorname{Leb}(U_n)^{\beta}$$

is finite for every $\beta \in [\beta_2, 1)$. Recall Corollary 3.10. From this point on, the proof of the proposition is similar to that of Proposition 2.3.

Since every cylinder U_n is a pre-image of a W_i' under some inverse branch, and the boundaries of the W_i' have limit capacity less than d, we may apply Proposition 4.1 to find $\rho > 0$ and C_0, depending only on the map and W_i', such that U_n admits a covering $\mathscr{B}(U_n)$ by balls of radius at least ρ^n such that

$$\sum_{B \in \mathscr{B}(U_n)} \operatorname{Leb}(B) \leqslant C_0 \operatorname{Leb}(U_n).$$

Then, given any $\gamma < 1$,

$$\begin{aligned}\sum_{B \in \mathscr{B}(U_n)} \operatorname{Leb}(B)^{\gamma} &\leqslant \sum_{B \in \mathscr{B}(U_n)} \operatorname{Leb}(B) \sup(\operatorname{Leb}(B)^{\gamma-1}) \\ &\leqslant C_0 \operatorname{Leb}(U_n)\, C_d^{\gamma-1} 2^{d(\gamma-1)} \rho^{nd(\gamma-1)}.\end{aligned}$$

Here we use, once more, the fact that $\operatorname{Leb}(B) \geqslant C_d \operatorname{diam}(B)^d$. Let $\mathscr{B}_n$ be the union of the $\mathscr{B}(U_n)$ over all $U_n \in \mathscr{U}_n$. Of course, $\mathscr{B}_n$ covers $\tilde{\Lambda}$. Observe also that the volume of n-cylinders of F decreases exponentially fast: there exists $c_2 > 0$ such that $\mathrm{Leb}(U_n) \leqslant e^{-c_2 n}$ for every U_n. That is because the map F is uniformly expanding, hence volume-expanding. Therefore, the previous inequality gives that

$$\sum_{B \in \mathscr{B}_n} \mathrm{Leb}(B)^\gamma \leqslant C_0 \sum_{U_n \in \mathscr{U}_n} \mathrm{Leb}(U_n)\, C_d^{\gamma-1} 2^{d(\gamma-1)} \rho^{nd(\gamma-1)}$$
$$\leqslant C_0\, C_d^{\gamma-1} 2^{d(\gamma-1)} \sum_{U_n \in \mathscr{U}_n} \mathrm{Leb}(U_n)^\beta\, e^{-c_2 n(1-\beta)} \rho^{nd(\gamma-1)}$$

for any $\beta < 1$. Fixing $\beta = \beta_2$, we get

$$\sum_{B \in \mathscr{B}_n} \mathrm{diam}(B)^{\gamma d} \leqslant C_d^{-\gamma} \sum_{B \in \mathscr{B}_n} \mathrm{Leb}(B)^\gamma \leqslant C_0\, C_d^{-1} 2^{d(\gamma-1)} S_0\, (e^{-c_2(1-\beta_2)} \rho^{d(\gamma-1)})^n.$$

Choose $\gamma_2 < 1$ close enough to 1 so that $e^{-c_2(1-\beta_2)} \rho^{d(\gamma_2-1)} < 1$. Then,

$$\sum_{B \in \mathscr{B}_n} \mathrm{diam}(B)^{\gamma d} \to 0 \qquad \text{as} \quad n \to \infty,$$

and so $m_{\gamma d}(\tilde{\Lambda}) = 0$, for every $\gamma_2 \leqslant \gamma < 1$. This implies $HD(\tilde{\Lambda}) \leqslant \gamma_2 d < d$. ▮

Since the Hausdorff dimension of a finite, or even countable, union of subsets is the supremum of their Hausdorff dimensions, see e.g., ref. 7, Chapter 4, these two propositions yield

$$\mathrm{HD}(\Lambda) = \mathrm{HD}(G \cup \tilde{\Lambda}) = \sup\{\mathrm{HD}(G), \mathrm{HD}(\tilde{\Lambda})\} < d.$$

The proof of Theorem 2 is complete.

ACKNOWLEDGMENTS

The authors are partially supported by CNPq 001/2000 and Pronex-Dynamical Systems. M.V. is also partially supported by Faperj.

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